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Rational solutions for Schwarzian integrable hierarchies

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Abstract. We give an approach to finding rational solutions of completely intagrable hierarchies, which makes use of the relationship between modifications and the Schwarzian equations obtained via the singular manifold method. This extends the recent work of Kudryashov, which allowed a simple derivation of the iteration used to construct sequences of such solutions. We also give a closed form for the index polynomial of the Schwarzian Korteweg–de Vries hierarchy.

In addition we consider the representation of rational solutions using lower families of the hierarchy. We give a simple representation under which the rational solutions remain solutions of every flow of the hierarchy. This representation also allows the inclusion of arbitrary data corresponding to negative indices.

We use our method to derive an alternative form of the Bäcklund transformation for the hierarchy of the second Painlevé equation, as well as new solutions of a hierarchy of breaking soliton equations. We also present here for the first time a Schwarzian version of this breaking soliton hierarchy.

1. Introduction

Rational solutions of integrable hierarchies have proved to be of interest to many workers over the years. The first to obtain such solutions for the Korteweg–de Vries (KdV) equation were Airault, McKean and Moser [1]; this was followed by the work of Adler and Moser [2], and Ablowitz and Satsuma [3, 4]. Weiss later showed how the singular manifold method [5], based on truncating the Weiss–Tabor–Carnevale (WTC) Painlevé expansion [6], could also be used to obtain rational solutions [7, 8] (for the KdV and Boussinesq equations, respectively). More recent approaches to this problem can be found in [9] and references therein.

The approach of Weiss involved finding and using invariances of the singular manifold equations in order to obtain a recursion relation for the iterative construction of rational solutions. In a recent paper [10] one of the authors of the present paper showed that this iterative formula could be much more simply obtained by considering, for modified equations, a double iteration of the Weiss truncation. In the current work we are interested in further extending this approach, and so also that of Weiss, in order to allow the construction of rational solutions for every member of a hierarchy of completely integrable partial differential equations (PDEs). For the KdV hierarchy, for example, this then provides an alternative means of deriving the solutions of Adler and Moser [2].

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The approach developed here involves understanding the connection between truncated Painlevé expansions and Miura maps, i.e. between the Schwarzian equations obtained via truncation and modified equations. The solutions we construct are in fact solutions of the corresponding Schwarzian hierarchies; solutions of the original hierarchy and others are then obtained using maps from the Schwarzian hierarchy. We also show how to derive a closed form for the index (resonance) polynomial of any family of any flow of the Schwarzian KdV (SKdV) hierarchy.

Such closed forms for the index polynomial of any family of any flow of a hierarchy are useful when it comes to understanding what happens when we use a truncated Painlevé expansion. They are also useful when we want to give an interpretation of the representation of rational solutions using 'lower' families, i.e. those whose standard Painlevé expansions have less than the full complement of arbitrary data. We later give a means of representing the rational solutions of the KdV hierarchy (simultaneously solutions of every flow) using such lower families; these representations include arbitrary data corresponding to negative indices.

The application of our approach to the hierarchy of the second Painlevé equation (P_{II}) [11–13] allows us to derive a new and very simple form of the Bäcklund transformation for that hierarchy [11]. This generalizes the results of Weiss [14] for P_{II} itself. Further new results are obtained through our consideration of a hierarchy of 'breaking soliton' equations in $2+1$ dimensions [15–17]. We give here for the first time a Schwarzian version of this hierarchy, and then use our approach to derive new solutions of this hierarchy. These solutions are rational in x and t and include arbitrary functions of y; suitable choices of these functions reduce these solutions to the rational solutions of the KdV hierarchy, which is a $(1+1)$-dimensional reduction of the $2+1$ hierarchy.

The paper is organized as follows. In section 2 we explore the relationship between Miura maps and truncated Painlevé expansions, and we show how the iterative formula for the generation of sequences of rational solutions can be derived from a double iteration of the constant-level truncation. In section 3 we obtain the families and index polynomial of the SKdV hierarchy. We also consider the result of seeking a truncated Painlevé expansion for the modified KdV (mKdV) hierarchy. A simple lemma is given which proves useful in section 4, which is where we derive rational solutions of the SKdV hierarchy. In section 5 we discuss the representation of solutions of the KdV (and SKdV) hierarchy using lower families. In section 6 we consider the application of our techniques to the P_{II} hierarchy. Section 7 sees the extension of our techniques to breaking soliton equations in $2+1$ dimensions. Section 8 is devoted to a summing up of our results and conclusions.

2. Muira maps and Painlevé truncation

Given two differential equations, say

$$D[u] = 0 \tag{1}$$

and

$$E[z] = 0 \tag{2}$$

we say that we have a map of equation (2) into (1) if we have a differential substitution of the form

$$u = F\left[z, z_x, \ldots\right] \tag{3}$$

such that for some differential operator $\widehat{C}$

$$D[u] = \widehat{C} E[z]. \tag{4}$$

Differential substitutions of the form (3) map solutions of (2) into solutions of (1); examples of such maps include Miura maps, linearizing transformations such as the Cole–Hopf transformation, and truncated Painlevé expansions. Here we will be interested in the connections between Miura maps and truncated Painlevé expansions, that is, between modified equations and singular manifold equations. We will see how such connections can be useful in deriving rational solutions.

Let us take the KdV hierarchy in the form

$$\omega_{t_{2n+1}} + \mathcal{R}^n \omega_x = \omega_{t_{2n+1}} + K_{2n+1}[\omega] = \omega_{t_{2n+1}} + \partial_x b^{n+1}[\omega] = 0 \qquad n = 0, 1, \ldots \tag{5}$$

where $\partial_x = \partial/\partial x$ (similarly in what follows for ∂_y, etc) and the recursion operator $\mathcal{R}$ is given [18] by

$$\mathcal{R} = \partial_x^2 + 4\omega + 2\omega_x \partial_x^{-1} \tag{6}$$

and let us consider the two successive modifications

$$\omega = \Omega[u] = u_x - u^2 \tag{7}$$

and

$$u = U[\varphi] = \frac{1}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right). \tag{8}$$

Then of course we have the well known relations

$$\mathcal{R}\Omega'[u]\Big|_{\omega=\Omega[u]} = \Omega'[u]\overline{\mathcal{R}} \tag{9}$$

where

$$\overline{\mathcal{R}} = \partial_x^2 - 4u^2 - 4u_x \partial_x^{-1} u \tag{10}$$

and

$$\overline{\mathcal{R}} U'[\varphi]\Big|_{u=U[\varphi]} = U'[\varphi]\tilde{\mathcal{R}} \tag{11}$$

where

$$\tilde{\mathcal{R}} = \varphi_x \partial_x^{-1} \varphi_x \partial_x \varphi_x^{-1} \partial_x \varphi_x^{-1} \partial_x \tag{12}$$

and $\Omega'[u]$ and $U'[\varphi]$ denote the Fréchet derivatives of $\Omega[u]$ and $U[\varphi]$ respectively:

$$\Omega'[u] = \partial_x - 2u \qquad U'[\varphi] = \tfrac{1}{2}\partial_x \varphi_x^{-1} \partial_x. \tag{13}$$

We then obtain the mKdV and SKdV hierarchies as

$$u_{t_{2n+1}} + \overline{\mathcal{R}}^n u_x = u_{t_{2n+1}} + \overline{K}_{2n+1}[u] = u_{t_{2n+1}} + \partial_x (\partial_x + 2u)\, b^n[u_x - u^2] = 0$$
$$n = 0, 1, \ldots \tag{14}$$

and

$$\varphi_{t_{2n+1}} + \tilde{\mathcal{R}}^n \varphi_x = \varphi_{t_{2n+1}} + \tilde{K}_{2n+1}[\varphi] = \varphi_{t_{2n+1}} + 2\varphi_x b^n\left[\tfrac{1}{2}\{\varphi; x\}\right] = 0 \qquad n = 0, 1, \ldots \tag{15}$$

respectively, where in (15) $\{\varphi; x\}$ denotes the Schwarzian derivative of φ

$$\{\varphi; x\} = \left(\frac{\varphi_{xx}}{\varphi_x}\right)_x - \frac{1}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 \tag{16}$$

introduced through the composition of (7) and (8)

$$\omega = u_x - u^2 = \tfrac{1}{2}\{\varphi; x\}. \tag{17}$$

Corresponding to equation (4) we of course have the relations

$$\omega_{t_{2n+1}} + K_{2n+1}[\omega] = \Omega'[u]\big(u_{t_{2n+1}} + \overline{K}_{2n+1}[u]\big) \tag{18}$$

and

$$u_{t_{2n+1}} + \overline{K}_{2n+1}[u] = U'[\varphi]\big(\varphi_{t_{2n+1}} + \tilde{K}_{2n+1}[\varphi]\big). \tag{19}$$

It is the SKdV hierarchy that we will use to generate rational solutions of the mKdV and KdV hierarchies. The mKdV and KdV hierarchies have the same sequence of singular manifold equations (see [19] for definitions) and for zero value of the spectral parameter this sequence becomes precisely the SKdV hierarchy (15). The connection between Miura maps and truncated Painlevé expansions can be seen using the invariance of the SKdV hierarchy under the Möbius group; sending $\varphi \to -1/\varphi$ gives

$$u = -\frac{\varphi_x}{\varphi} + \frac{1}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right) \tag{20}$$

is a solution of the mKdV hierarchy provided that equation (15) holds. Given that (8) is also a solution of the mKdV hierarchy we see that (20) defines an auto-Bäcklund transformation between two solutions u and u_1 of the mKdV hierarchy as

$$u = -\frac{\varphi_x}{\varphi} + u_1 \qquad u_1 = \frac{1}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right). \tag{21}$$

The invariance of the mKdV hierarchy under $u \to -u$ then gives a second auto-Bäcklund transformation between two solutions v and v_1 as

$$v = \frac{z_x}{z} + v_1 \qquad v_1 = -\frac{1}{2}\left(\frac{z_{xx}}{z_x}\right) \tag{22}$$

for any solution z of

$$S_{2n+1}[z] \equiv z_{t_{2n+1}} + 2z_x b^n\left[\tfrac{1}{2}\{z; x\}\right] = 0. \tag{23}$$

Expressions (21) and (22) are of course the Painlevé expansions for the principal families of the mKdV hierarchy truncated at constant level. From equation (19) we find that corresponding to (20) we have the relation

$$u_{t_{2n+1}} + \overline{K}_{2n+1}[u] = \frac{1}{2}\partial_x\left(\frac{1}{\varphi_x}\partial_x - \frac{2}{\varphi}\right)S_{2n+1}[\varphi] \tag{24}$$

which gives precisely the same result as substituting the truncated Painlevé expansion (21) in the mKdV hierarchy (see section 3), and similarly for the truncation (22).

Following the approach in [10] we perform a double iteration of this constant-level truncation and identify $u_1 = v$ to obtain

$$\varphi_x = \frac{z^2}{z_x} \tag{25}$$

which is the formula used by Weiss to obtain rational solutions of the KdV equation. This formula allows the iterative construction of solutions of (23) via

$$z_{k+1,x} = \frac{z_k^2}{z_{k,x}}. \tag{26}$$

This double iteration of the Weiss auto-Bäcklund transformation makes use of the two principal families of the mKdV hierarchy and greatly simplifies the derivation of the iterative formula (26). Being a double iteration of the constant-level truncation, this approach is different from those based on the use of two singular manifolds [20, 21]; recall that the equations that we are using here are the singular manifold equations of the mKdV hierarchy for zero value of the spectral parameter, i.e. they are those found using the Weiss constant-level truncation.

3. Families and indices of the SKdV hierarchy

In [22] we gave a closed form for the Painlevé index polynomial of any family of any flow of the KdV hierarchy; the proof was presented in [23]. We then showed [24] how the Muira map (7) could be used to obtain the families and index polynomial for the mKdV hierarchy. We now use the Miura map (8) to obtain the families and index polynomial for the SKdV hierarchy (15). Use will be made of these results later.

Let us recall that in the application of the WTC Painlevé test to the KdV hierarchy (5), the families are given [22–26] as

$$\omega = \omega_0 z^{-2} + \cdots \tag{27}$$

where

$$\omega_0 = -k(k+1)z_x^2 \qquad k = 1, \ldots, n \tag{28}$$

and that the index polynomial for the kth family [22–24] is

$$Q_{2n+1}(r;k) = (r-2n-2)\prod_{i=1}^{k}(r+2i-1)(r-2i-2n-2) \times \prod_{j=k+1}^{n}(r-2j-1)(r+2j-2n-2). \tag{29}$$

For the mKdV hierarchy (14) we have the families [24]

$$u = u_0 z^{-1} + \cdots \tag{30}$$

with

$$u_0 = \pm k z_x \qquad k = 1, \ldots, n \tag{31}$$

and the kth family having the index polynomial [24]

$$\overline{Q}_{2n+1}(r;k) = (r-2n-1)\prod_{i=1}^{k}(r+2i-1)(r-2i-2n) \times \prod_{j=k+1}^{n}(r-2j+1)(r+2j-2n-2). \tag{32}$$

Note that the principal family ($k = 1$) has positive indices $2, 3, \ldots, 2n-1$, and $2n+1$, $2n+2$. This then allows us to understand what the identity (24) tells us about the result of substituting the truncated Painlevé expansion (21) in the mKdV hierarchy (14). The leading-order behaviour of (14) is $\varphi^{-(2n+2)}$, and the coefficient of this term vanishes due to our choice of leading-order coefficient. At $\varphi^{-(2n+1)}$, we find u_1 is determined as in (21). Then the coefficients of $\varphi^{-2n}, \varphi^{-2n+1}, \ldots, \varphi^{-3}$ all vanish, since these correspond to the indices $2, 3, \ldots, 2n-1$. The coefficient of φ^{-2}, which does not correspond to an index, then provides the constraint $S_{2n+1}[\varphi] = 0$ (see equation (24)). The coefficients of φ^{-1} and

φ^0, corresponding to the indices $2n+1$ and $2n+2$, then vanish as a consequence of this constraint; this is what we would usually expect with a truncated Painlevé expansion. All of this follows from the identity (24).

We now find the families and index polynomial of the SKdV hierarchy (15). Considering the leading-order terms of the Miura map (8) we see that for any family

$$\varphi = \varphi_0 z^p + \cdots \tag{33}$$

of the SKdV hierarchy (15) we have

$$u_0 = \tfrac{1}{2}(p-1)z_x \tag{34}$$

and so

$$p = \pm 2k+1 \qquad k = 0, \ldots, n \tag{35}$$

where in (35) we allow k to run from 0 to n since $u_0 = 0$ is a solution of the polynomial in u_0 obtained at leading order from the substitution of (30) into the mKdV hierarchy (14). We thus obtain the leading-order exponents

$$p = \pm(2k-1) \qquad k = 1, \ldots, n \tag{36}$$

and in addition

$$p = (2n+1). \tag{37}$$

However, this last is not a choice of leading-order exponent of the hierarchy (15) but rather is due to the action of the operator $U'(\varphi)$ in (19). We thus find that the hierarchy (15) has the families (33) with p given by (36). Also of course we have the regular solution with $p = 0$, but we do not consider this here. Furthermore, the invariance of the SKdV hierarchy under the action of the Möbius group tells us that, in determining the indices, we only need to consider the choices

$$p = -(2k-1) \qquad k = 1, \ldots, n \tag{38}$$

corresponding to which we have $u_0 = -kz_x$, $k = 1, \ldots, n$. Considering the dominant terms of (19), and recalling the construction of the index polynomial [22–24], we obtain

$$\begin{aligned} &\overline{K}'_{2n+1}[-kz_x z^{-1}]U'[\varphi_0 z^{-(2k-1)}]z^{r-(2k-1)}\Big|_{z_x-1=\varphi_{0,x}=0} \\ &\qquad = U'[\varphi_0 z^{-(2k-1)}]\tilde{K}'_{2n+1}[\varphi_0 z^{-(2k-1)}]z^{r-(2k-1)}\Big|_{z_x-1=\varphi_{0,x}=0} \end{aligned} \tag{39}$$

which then gives

$$\tilde{Q}_{2n+1}(r;k) = \frac{r(r-2k+1)}{(r-2n-2k)(r-2n-1)}\overline{Q}_{2n+1}(r;k) \tag{40}$$

and thus

$$\tilde{Q}_{2n+1}(r;k) = (r+1)\prod_{i=1}^{k-1}(r+2i+1)(r-2i-2n)\prod_{j=k}^{n}(r-2j+1)(r+2j-2n). \tag{41}$$

This is the index polynomial for the kth family (36) of the nth flow of the SKdV hierarchy (15).

It is clear from the leading-order behaviours (36), and the fact that the SKdV hierarchy (15) involves only derivatives of φ, that the quantities

$$\sigma_k = x^{2k+1} + A_k \qquad k = 0, 1, 2, \ldots \tag{42}$$

where each A_k is constant, are solutions of the t_{2n+1}-flows of the SKdV hierarchy for every $n > k$. We can also give the following lemma, which will be of practical use later.

Lemma. The functions τ_k defined by

$$\tau_k = x^{2k+1} + B_k \qquad k = 0, 1, 2 \ldots \tag{43}$$

where B_k is a function of all the flow times, are solutions of the SKdV hierarchy for every $n \geqslant k$ provided that

$$\frac{\partial B_k}{\partial t_{2n+1}} = 0 \qquad n > k \tag{44}$$

and (for $n = k$)

$$\frac{\partial B_k}{\partial t_{2k+1}} = (-1)^{k+1}(2k+1)2^k \prod_{l=0}^{k-1} \frac{(2l+1)\,[k\,(k+1) - l\,(l+1)]}{(l+1)}. \tag{45}$$

There are no solutions of the SKdV hierarchy of the form (43) for $n < k$.

Proof. Noting that

$$2b^1\left[\frac{1}{2}\{\tau_k; x\}\right] = \{\tau_k; x\} = -\frac{2k(k+1)}{x^2} \tag{46}$$

and using the recursion relation for b^n, we can show by induction that

$$2b^n\left[\frac{1}{2}\{\tau_k; x\}\right] = (-1)^n \frac{2^n}{x^{2n}} \prod_{l=0}^{n-1} \frac{(2l+1)\,[k\,(k+1) - l\,(l+1)]}{(l+1)}. \tag{47}$$

This vanishes for $n > k$, since (43) corresponds to a leading-order behaviour of the t_{2n+1} flow for $k = 0, 1, \ldots, n-1$. It then follows that substitution of (43) in the t_{2n+1} flow, $n > k$, yields the condition (44). In the case $n = k$ we substitute (43) in the t_{2k+1} flow to obtain (45). It is clear that there are no solutions of the t_{2n+1} flows of the form (43) for $n < k$ since we then obtain that $2\tau_{k,x} b^n[\frac{1}{2}\{\tau_k; x\}]$ does not vanish and is proportional to $x^{2(k-n)}$, and so cannot balance $\partial B_k/\partial t_{2n+1}$. Of course these flows can still have rational solutions of weight $2k+1$. □

4. Rational solutions of the SKdV hierarchy

Here we use the results of the previous two sections to derive rational solutions of the SKdV hierarchy; rational solutions of the mKdV and KdV hierarchies are then obtained using (Miura) maps. We seek rational solutions of the entire hierarchies, i.e. depending on all of the flow times. In this way we generalize the results of Weiss [7] and recent results of Kudryashov [10]. Our approach also has the advantage over that of Weiss in that we obtain the recursion relation (26) directly, instead of having to first investigate symmetries of the Schwarzian hierarchy. For the KdV hierarchy the rational solutions we find are of course those of Adler and Moser [2]; we later use these to shed light on the role played by lower families in Painlevé analysis, and also on that played by negative indices. Later sections will also see us extend the approach detailed here first to hierarchies of ODEs, and then to $2+1$ hierarchies of 'breaking soliton' equations.

We derive rational solutions of the SKdV hierarchy (23) using the iterative formula (26)

$$z_{k+1,x} = \frac{z_k^2}{z_{k,x}} \tag{48}$$

and beginning with the trivial solution

$$z_0 = x. \tag{49}$$

At each step we introduce a function of integration $D_k(t_3, t_5, t_7, \ldots)$ in z_k, and we also renormalize so that z_k is the quotient of two monic polynonials. The dependence of D_k on the flow times $t_3, t_5, t_7, \ldots$ is determined by substitution of the resulting solution into the SKdV hierarchy. Once we have determined the dependence of z_k on t_{2n+1}, $n < k$, the above lemma can be very useful for fixing its dependence on t_{2n+1} for $n \geqslant k$ (see below). Without loss of generality, we do not include constants of integration when solving for D_k.

From equation (48) we obtain $z_1 = x^3 + D_1$, and from our lemma we see that we must have

$$z_1 = x^3 + 12t_3. \tag{50}$$

Iterating again yields $z_2 = x^5 + 60x^2t_3 - 720t_3^2x^{-1} + D_2$. Substitution into the t_3-flow gives that $\partial D_2/\partial t_3 = 0$. We can then set $t_3 = 0$ in z_2 and use our lemma to obtain the condition that in order to be a common solution of the entire SKdV hierarchy we must have $D_2 = -720t_5$, and so

$$z_2 = \frac{1}{x}(x^6 + 60x^3t_3 - 720t_3^2 - 720xt_5). \tag{51}$$

A third iteration gives

$$z_3 = \frac{1}{x^3 + 12t_3}\big(x^{10} + 180x^7t_3 - 5040x^5t_5 + 302\,400x^2t_3t_5 + 302\,400xt_3^3 - 1\,209\,600t_5^2\big) + D_3$$

and substitution into the t_3 and t_5 flows gives $\partial D_3/\partial t_3 = 0$ and $\partial D_3/\partial t_5 = 0$. Then we can set $t_3 = t_5 = 0$ in z_3 and use our lemma to obtain $D_3 = 100\,800t_7$. Thus

$$z_3 = \frac{1}{x^3 + 12t_3}\big(x^{10} + 180x^7t_3 - 5040x^5t_5 + 100\,800x^3t_7 + 302\,400x^2t_3t_5 + 302\,400xt_3^3 + 1\,209\,600t_3t_7 - 1\,209\,600t_5^2\big). \tag{52}$$

A fourth iteration gives

$$z_4 = D_4 + 63\int^x \frac{z_3^2}{z_{3,x}}\,\mathrm{d}x \tag{53}$$

where 63 is our normalizing coefficient. Substitution into the t_3 flow gives

$$\frac{\partial D_4}{\partial t_3} = 6\,350\,400t_3^2 \tag{54}$$

and so

$$D_4 = 2\,116\,800t_3^3 + \tilde{D}_4 \tag{55}$$

for some function $\tilde{D}_4(t_5, t_7, \ldots)$. Substitution into the t_5 and t_7 flows then gives

$$\frac{\partial \tilde{D}_4}{\partial t_5} = 0 \tag{56}$$

and

$$\frac{\partial \tilde{D}_4}{\partial t_7} = 0. \tag{57}$$

We can now set $t_3 = 0$, $t_5 = 0$ and $t_7 = 0$ in z_4 and use our lemma to obtain

$$\tilde{D}_4 = -25\,401\,600t_9. \tag{58}$$

This then gives

$$
\begin{aligned}
z_4 = {} & \frac{1}{x^6 + 60x^3t_3 - 720t_3^2 - 720xt_5}\big(x^{15} + 420x^{12}t_3 - 20\,160x^{10}t_5 + 25\,200x^9t_3^2 \\
& + 907\,200x^8t_7 + 907\,200x^7t_3t_5 + 2\,116\,800x^6t_3^3 - 25\,401\,600x^6t_9 \\
& - 76\,204\,800x^5t_3t_7 - 76\,204\,800x^5t_5^2 - 381\,024\,000x^4t_3^2t_5 \\
& - 254\,016\,000x^3t_3^4 - 1\,524\,096\,000x^3t_3t_9 + 1\,524\,096\,000x^3t_5t_7 \\
& - 4\,572\,288\,000x^2t_3^2t_7 + 4\,572\,288\,000x^2t_3t_5^2 - 1\,524\,096\,000xt_3^3t_5 \\
& - 18\,289\,152\,000xt_7^2 + 18\,289\,152\,000xt_5t_9 - 1\,524\,096\,000t_3^5 \\
& + 18\,289\,152\,000t_3^2t_9 - 36\,578\,304\,000t_3t_5t_7 + 18\,289\,152\,000t_5^3\big). \qquad (59)
\end{aligned}
$$

In this way we construct rational solutions of the entire SKdV hierarchy, and thus also of the mKdV and KdV hierarchies. This provides an alternative approach to that of Adler and Moser [2]. These solutions include as special cases solutions of the form (43), solutions of the t_{2n+1} flows for $n \geqslant k$. We will see later that the above solutions can also be obtained as a reduction of those for a breaking soliton hierarchy. In addition, appropriate similarity reductions of the above lead to rational solutions of the $\mathrm{P_{II}}$ hierarchy. We now consider the representation of the above solutions using lower families of the hierarchy, and the role played by negative indices in such representations.

5. Representing rational solutions using lower families

In the previous section we obtained rational solutions of the SKdV hierarchy, and thus also of the mKdV and KdV hierarchies. The rational solutions of the SKdV hierarchy can all be written [7] in the form

$$z_k = \frac{P_{k+1}}{P_{k-1}}. \qquad (60)$$

Rational solutions of the KdV hierarchy are then obtained [2, 7] as

$$\omega = 2(\log P_k)_{xx} \qquad k = 1, 2, \ldots \qquad (61)$$

where the recursion relation for the polynomials $P_k(x, t_3, t_5, \ldots)$ is as given in [2, 7].

In this section we discuss the representation of these rational solutions using the so-called 'lower' families of the KdV hierarchy. Of particular interest is the role played by negative indices in such representations. We recall from section 3 that the kth family of the KdV hierarchy has k negative indices, $r = -(2i - 1)$, $i = 1, \ldots, k$. These lower families are often regarded as secondary in some sense; we believe that from many points of view they are in fact just as important as principal families. We note that a thorough discussion of the singularites of the Kadomtsev–Petviashvili hierarchy was undertaken in [27]. However, the role played by negative indices does not seem to have been addressed there.

We briefly recall the comments of Newell *et al* [25] that each P_k is a weighted polynomial of degree $k(k+1)/2$, and setting $t_3 = t_5 = t_7 = \ldots = 0$ in (61) yields

$$\omega = -k(k+1)x^{-2} \qquad (62)$$

i.e. we obtain the leading-order behaviours for the lower families of the KdV hierarchy. Thus the kth rational solution is seen to unfold the singularity near the coalescence of

$k(k+1)/2$ poles; the fact that equation (62) is no longer a solution of the entire KdV hierarchy is explained by saying that since the Painlevé analysis is local, 'it does not particularly care'. This interpretation of lower families as giving a representation of solutions near a coalescence of singularities is clearly one which requires further, and more careful, investigation. In what follows we show how these lower families can be used to provide a representation of the rational solutions; using this representation they remain solutions of the entire hierarchy. We will also see that this representation allows the inclusion of arbitrary data corresponding to negative indices.

We illustrate our results using the first three rational solutions of the KdV hierarchy. The associated polynomials are:

$$P_1 = x \tag{63}$$

$$P_2 = x^3 + 12t_3 \tag{64}$$

$$P_3 = x^6 + 60x^3t_3 - 720t_3^2 - 720xt_5. \tag{65}$$

We recall that the corresponding rational solutions are solutions of the entire KdV hierarchy.

The first of these of course gives $\omega = -2x^{-2}$, which corresponds to the pole expansion for the first (and principal) family of every member of the hierarchy. Taking the second rational solution, we expand in descending powers of x to obtain

$$\omega = 2(\log P_2)_{xx} = -6x^{-2} + 288t_3x^{-5} - 6048t_3^2x^{-8} + \cdots \tag{66}$$

and thus we see that we are able to use the second family to provide a representation of this rational solution. For $n > 1$, for which such a family exists, t_3 is a constant of the t_{2n+1} flow; this constant represents arbitrary data introduced at x^{-5}, and so corresponds to the index at $r = -3$. For these flows, setting $t_3 = 0$ means making a special choice of a constant in the solution and thus yields the particular solution $\omega = -6x^{-2}$. Of course this last is no longer a solution of the t_3 flow since setting $t_3 = 0$ interferes with the flow time of this member of the hierarchy. However, the expansion (66) is of course a solution of every member of the hierarchy, including the t_3 flow.

Expanding the third rational solution in descending powers of x gives

$$\omega = 2(\log P_3)_{xx} = -12x^{-2} + 1440t_3x^{-5} - 43\,200t_5x^{-7} + \cdots \tag{67}$$

and so we obtain a representation of this solution using the third family of the KdV hierarchy. For $n > 2$ this is a family of the t_{2n+1} flow, and we have corresponding to the indices at $r = -3$ and $r = -5$ arbitrary constants t_3 and t_5. For these flows setting $t_3 = t_5 = 0$ then gives the particular solution $\omega = -12x^{-2}$. However, this is no longer a solution of the t_3 and t_5 flows, because we have interfered with the flow times. Again the expansion (67) remains a solution of every member of the hierarchy, including the t_3 and t_5 flows.

In general, the kth rational solution has an expansion in descending powers of x that can be identified with the kth family of the KdV hierarchy, with arbitrary data corresponding to the negative indices $r = -(2i-1)$, $i = 2, \ldots, k$. In this way we obtain a representation for the full rational solutions, and not just for the particular cases (62). We recall also the results of [26, 22], where a connection was established between negative resonances and the presence of lower order commuting flows. We note that arbitrary data at $r = -1$ is trivially included in the above descending series solutions by shifting x and then re-expanding.

We note from (41) that the negative index structure of the families of the SKdV hierarchy is the same as that of the families of the mKdV and KdV hierarchies. We can see from equations (50), (51), (52) and (59) that the representation of z_1, z_2, z_3 and z_4 by descending series solutions will lead to similar conclusions to those presented above for the KdV hierarchy.

6. Application to ODEs: the P_{II} hierarchy

In this section we consider very briefly the application of our approach to hierarchies of ODEs, namely the P_{II} hierarchy [11–13] and a Schwarzian P_{II} hierarchy introduced here. This allows us to derive a new alternative form for the Bäcklund transformation for the P_{II} hierarchy. We begin with the SKdV hierarchy (15), and we make the similarity reduction

$$\varphi(x, t_{2n+1}) = [(2n+1)t_{2n+1}]^{m_n/(2n+1)}\psi(X) \qquad X = x[(2n+1)t_{2n+1}]^{-1/(2n+1)} \tag{68}$$

where m_n is a constant; this then yields the Schwarzian hierarchy

$$m_n\psi - X\psi' + 2\psi' b^n\left[\tfrac{1}{2}\{\psi; X\}\right] = 0. \tag{69}$$

We note that with $m_n = 0$ and the substitution $F = \frac{1}{2}\{\psi; X\}$, this last is the sequence of first Painlevé equations of higher order given in [13].

Under the action of the operator $(\psi')^{-1}\mathrm{d}/\mathrm{d}X$, the sequence of ODEs (69) becomes

$$\left(\frac{\mathrm{d}}{\mathrm{d}X} + 2V\right) b^n[V' - V^2] - XV - \alpha_n = 0 \tag{70}$$

where we have made the substitution $V = (\psi'')/(2\psi')$, and we have set $\alpha_n = (1 - m_n)/2$. The sequence of ODEs (70) is the hierarchy of higher-order P_{II} equations [11–13]. We now consider the iteration which corresponds to our iteration $\varphi_k \to \varphi_{k+1}$ of the SKdV hierarchy. We take

$$\varphi_k = [(2n+1)t_{2n+1}]^{m_n/(2n+1)}\psi_k(X) \tag{71}$$

and

$$\varphi_{k+1} = [(2n+1)t_{2n+1}]^{M_n/(2n+1)}\psi_{k+1}(X). \tag{72}$$

The iterative formula

$$\varphi_{k+1,x} = \frac{\varphi_k^2}{\varphi_{k,x}} \tag{73}$$

then yields

$$M_n = m_n + 2 \qquad \text{and} \qquad \psi'_{k+1} = \frac{\psi_k^2}{\psi'_k}. \tag{74}$$

Note that $M_n = m_n + 2$ corresponds to $\alpha_n^{\text{new}} = \alpha_n^{\text{old}} - 1$. This shift on α_n can be understood by recalling that the iteration (73) was derived by sending $\varphi \to -1/\varphi$ in (8) to obtain (21), and then sending $u \to -u$ to obtain (22); mirroring this sequence of transformations for ψ and V then gives the above shift on α_n. Thus we have derived a very simple (although somewhat implicit) form for the Bäcklund transformation for the P_{II} hierarchy [11]. This shift on α_n and the mapping (74) were in fact obtained for P_{II} itself in [14]. However, the approach in [14] involved the direct consideration of additional symmetries of (69) (for $n = 1$).

A derivation of the Bäcklund transformation for the P_{II} hierarchy (in the form given in [11]) can be found in [28]; this derivation uses a new extension of the Painlevé truncation approach. For the special case $n = 2$, the Bäcklund transformation in the form presented in [11] has also been given in [29] and [30].

Beginning with the solution $\psi = X$ for $m_n = 1$ (corresponding to $V = 0$ for $\alpha_n = 0$), and using also the symmetry $(V, \alpha_n) \to (-V, -\alpha_n)$, we are able to use the iteration (74) to construct rational solutions for the P_{II} hierarchy. Taking each equation separately, we obtain a sequence of solutions for that equation, one for every integer value of α_n. At each step we again renormalize so that ψ is the quotient of two monic polynonials. Also at each step we introduce a constant of integration which is fixed by substitution back into (69). The resulting rational solutions can all be found by appropriate similarity reduction of the rational solutions derived earlier for the SKdV and mKdV hierarchies, and so we do not give them explicitly here.

7. A 2+1 Schwarzian breaking soliton hierarchy

In this section we apply our approach to a hierarchy of 'breaking soliton' equations in $2+1$ dimensions [15–17]

$$\omega_{t_{2n+1}} + \mathcal{R}^n \omega_y = 0 \qquad n = 0, 1, \ldots \tag{75}$$

where $\mathcal{R}$ is as in (6), which has as its first non-trivial flow an equation due to Calogero [15] (discussed in detail by Bogoyavlenskii [31]; see also Schiff [32])

$$\omega_{t_3} + \omega_{xxy} + 4\omega\omega_y + 2\omega_x \partial_x^{-1} \omega_y = 0. \tag{76}$$

From equations (9) and (11) it is clear that the Miura maps (7) and (8) respectively give a modified breaking soliton hierarchy

$$u_{t_{2n+1}} + \overline{\mathcal{R}}^n u_y = 0 \qquad n = 0, 1, \ldots \tag{77}$$

where $\overline{\mathcal{R}}$ is as in (10), and a Schwarzian breaking soliton hierarchy

$$\varphi_{t_{2n+1}} + \tilde{\mathcal{R}}^n \varphi_y = 0 \qquad n = 0, 1, \ldots \tag{78}$$

where $\tilde{\mathcal{R}}$ is as in (12). We note that the hierarchy (77) does not seem to have been written down before, although the first non-trivial member

$$u_{t_3} + u_{xxy} - 4u^2 u_y - 4u_x \partial_x^{-1}(uu_y) = 0 \tag{79}$$

does appear in [31]. The hierarchy (78) also appears to be new; its first non-trivial flow can be written as

$$\varphi_{t_3} + \varphi_x \partial_x^{-1} \partial_y \{\varphi; x\} = 0. \tag{80}$$

This last is the singular manifold equation of both (76) and (79) for zero value of the spectral parameter $\lambda(y, t)$ appearing in the corresponding non-isospectral scattering problems.

It is easy to see that equation (77) is invariant under $u \to -u$. We can also show formally that the hierarchy (78) is invariant under the action of the Möbius group. We first rewrite the hierarchy (78) as

$$E_{2n+1}[\varphi] \equiv \varphi_{t_{2n+1}} + \varphi_x F_{2n+1}[\varphi] = 0 \tag{81}$$

where

$$F_{2n+1}[\varphi] = T^{n-1} F_3[\varphi] \tag{82}$$

and T is the operator

$$T = \partial_x^{-1} \varphi_x \partial_x \varphi_x^{-1} \partial_x \varphi_x^{-1} \partial_x \varphi_x \tag{83}$$

and

$$F_3 = \partial_x^{-1} \partial_y \{\varphi; x\}. \tag{84}$$

It is then simple matter to show that if F is invariant under the Möbius group then so is TF (up to the non-locality due to the integration in T). Then since F_3 is invariant so is any F_{2n+1} and thus any member of the hierarchy $E_{2n+1} = 0$.

We now apply our approach to the breaking soliton hierarchies given above. We give our results in terms of solutions of the hierarchy (75). These solutions are all new. For reasons of space we confine ourselves here to considering only the first three common solutions. At each step we again renormalize and introduce a function of integration whose dependence on the flow times is determined by substitution into the hierarchy. These functions of integration depend also on y and so the solutions we obtain are rational in x and t but also involve arbitrary functions of y. Special choices of these functions allow us to make a reduction of the solutions obtained to those obtained earlier for the KdV hierarchy.

For the hierarchy (75), the solutions we obtain can be written as

$$\omega = 2\,(\log P_k)_{xx}\,. \tag{85}$$

The first three polynomials P_k we obtain are

$$P_1 = x + f(y) \tag{86}$$

$$P_2 = [x + f(y)]^3 + 12[f'(y)t_3 + g(y)] \tag{87}$$

$$\begin{aligned} P_3 = {} & [x + f(y)]^6 + 60[x + f(y)]^3[f'(y)t_3 + g(y)] - 720[f'(y)t_3 + g(y)]^2 \\ & +360[x + f(y)][f''(y)t_3 + 2g'(y)]t_3 \\ & - 720[x + f(y)][f'(y)t_5 + h(y)] \end{aligned} \tag{88}$$

where $f(y)$, $g(y)$ and $h(y)$ are arbitrary functions of y. The KdV hierarchy arises as the reduction $\partial_y = \partial_x$ of (75) and accordingly the solutions of the KdV hierarchy corresponding to (61) and (63), (64), (65) are obtained from equations (85) and (86), (87), (88) above by choosing $f(y) = y$, and $g(y)$, $h(y)$ both constant. It is clear that our earlier results on negative indices and the representation of solutions using lower families hold for these breaking soliton hierarchies also.

8. Conclusions

The recent work of Kudryashov simplified the earlier approach of Weiss to the question of obtaining rational solutions. Here, by exploring the link between Miura maps and truncated Painlevé expansions, we have extended this work of Kudryashov, and thus also that of Weiss, in order to derive rational solutions of every member (simultaneously) of a hierarchy. This then gives an alternative approach, based on the use of corresponding Schwarzian hierarchies, to the construction of such solutions.

We have used our knowledge of the indices for every family of every flow of the mKdV hierarchy to understand what our mapping from the SKdV hierarchy to the mKdV hierarchy is telling us about the results of using a truncated Painlevé expansion for the mKdV hierarchy. We have also obtained a closed form for the index polynomial of every family of every flow of the SKdV hierarchy.

We have given a representation of the rational solutions obtained for the KdV hierarchy using lower families of the hierarchy. Using this representation allows the rational solutions to remain solutions of every flow of the hierarchy. Also in this representation we have seen the inclusion of arbitrary data corresponding to negative indices.

We have used the approach outlined here to derive an alternative form for the Bäcklund transformation for the P_{II} hierarchy, and also to find new solutions for a hierarchy of breaking soliton equations in $2 + 1$ dimensions. We have also given for the first time a Schwarzian version of this breaking soliton hierarchy.

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